Limited Observation in Mutual Consent Networks

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Abstract

This paper studies mutual consent social networks in which individuals imperfectly monitor others' network ties and have incomplete information about the benefits of network participation. I introduce the *Conjectural Pairwise Stability* concept, which generalizes Jackson and Wolinsky's (1996) *Pairwise Stability* concept to allow for limited observation, and apply it to a specific mutual consent network formation game. While limited observation generally leads to the existence of less efficient stable networks, I find that it can also lead to the existence of efficient stable networks. Moreover, stability restrictions considered in previous work lose their refining power as observation becomes more limited.

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1 Introduction

Social networks underlie many economic and social activities, such as the spread of valuable information or the trade of goods and services.[1] These social networks generally have two common features. The first feature, that network ties usually require mutual consent, is widely acknowledged in game theoretic studies of social networks [e.g., Jackson (2003)]. Examples of social ties that require mutual consent include friendship networks and business connections. On the contrary, the second feature, that an individual usually only observes a local area of her network, has received very little attention in the game theoretic literature. This is true despite the empirical work (largely by sociologists) which shows that individuals, in general, only observe their own direct ties and the ties of their network neighbors [Laumann (1969), Friedkin (1983), Kumbasar, Romney, and Batchelder (1994), Bondonio (1998), and Casciaro (1998)].

Since social networks exist in abundance, limited observation obviously does not prevent them from forming, nor does it prevent individuals from benefiting from others' network ties even when not aware of them. It might, however, lead to the persistence of networks that would not persist under full information. Consider the following example. Unemployed Amy learns of a job opening from her friend Barry, who learned of the job from his cousin Clint, who just happened to learn of the job from his neighbor Dinah. If Amy does not observe Clint interact with Dinah, then she might not know the origin of the job news. Of course, even if Amy knew that Clint's news came from Dinah, Amy still might not know what other job news Dinah has. In game theoretic terminology, Amy *imperfectly monitors* Clint's links and has *incomplete information* about the potential value of linking to Dinah. Although Amy benefits from her connections despite her limited information, she is possibly better off if she formed a tie directly with Dinah, since Dinah may have other job news that did not work its way through Clint and Barry. In fact, if Amy knew more about Dinah, such as whether Dinah works in a particular industry, then Amy might prefer to communicate

[1] Examples and references are numerous, e.g., see Conley and Udry (2001) on learning about production techniques through informal networks in developing countries, Calvó-Armengol and Jackson (2004) on learning about job openings through contacts, and Kranton and Minehart (2001) on buyer-seller networks.

directly with Dinah instead of indirectly, but without knowing anything else about Dinah, Amy may decide to not initiate such communication. An inefficient network might persist because of Amy's limited information.

This paper presents a game theoretic examination of such settings. I first introduce an original network stability concept, *Conjectural Pairwise Stability* (CPS), for the study of limited observation in mutual consent networks. CPS generalizes the *Pairwise Stability* (PS) concept introduced by Jackson and Wolinsky (1996). A network is PS if no single link created by a pair of players improves each deviator's utility, and if any removal of a link by a player reduces her utility. Implicit in the PS concept is that each individual has full information about the change in her utility arising from any single link deviation, and, as such, the PS concept is not appropriate if individuals have limited information about the network. The CPS concept generalizes PS by relaxing the restriction that individuals' beliefs about the network must be correct. More precisely, the CPS concept allows for networks to be stable when each player believes that any single link deviation is utility decreasing and when each's beliefs are not contradicted by available information about the network. Thus, the CPS concept allows players to have incorrect conjectures in stable networks, so long as they have no information to contradict those conjectures.

After presenting some basic network notation in Section 2, Section 3 formally defines CPS and relates it to PS. Section 4 then applies the CPS concept to the *connections model* first studied by Jackson and Wolinsky (1996) and later extended by Johnson and Gilles (2000) and Jackson and Rogers (2005) to include heterogeneity in link benefits and costs. As in the connections model studied by Johnson and Gilles (2000), I allow for heterogeneity in the benefits of forming links. However, unlike previous work, I also allow for imperfect monitoring of others' links and incomplete information about the benefits of forming links. To formally characterize this informational structure, I use the x/y-link observation information structure introduced by McBride (2005). Each i observes network ties that are x or fewer links away from her, and observes the player types of those players within $y \leq x$ links from her. This concept mimics the limited observation present in actual networks and allows me to examine how the set of network equilibria changes as players' observational capabilities (x or y) increase or decrease.

Section 4 presents two new results. First, limited network observation in a mutual consent setting leads to the persistence of inefficient networks that would not persist under full observation. This result compares directly with a similar result for non-cooperative networks [McBride (2005, 2006)], thereby showing that the primary impact of limited observation is robust to the network formation setting (i.e., whether non-cooperative or mutual consent). Second, limited observation, in some settings, actually leads to the existence of stable networks that are more efficient than those under full observation. In effect, an individual's limited observation may lead her to unknowingly choose actions that are individually detrimental but socially beneficial. This result contrasts with the findings in McBride (2005, 2006) in which limited observation only hinders efficiency.

Because the CPS concept imposes a minimal set of conditions for stability, Section 5 investigates the impacts of two refinements that have been considered in the networks literature: common knowledge of rationality and coalitional deviations. This paper's third new result is that the refining power of each refinement diminishes as players' observation becomes more limited. Under certain conditions, imposing common knowledge of rationality does not change the set of CPS networks when observation is very limited, but it does change the set of CPS under higher levels of observation. This finding relates directly to Rubinstein and Wolinsky's (1994) *Rationalizable Conjectural Equilibrium* refinement of *Conjectural Equilibrium* for non-cooperative games with imperfect monitoring. Similarly, the refining power that comes by considering coalitional deviations is greater under higher levels of observation. In short, natural refinements of the CPS concept do not greatly refine the set of CPS networks when individuals have severely limited network information.

The general contribution of this paper is the introduction and application of the CPS concept to the study of mutual consent network formation games with imperfect monitoring and incomplete information. This concept can be applied to other mutual consent network games, thereby allowing researchers to study the relationship between stability, efficiency, and observation in other mutual consent network settings. The more specific contribution of this paper relates to the new findings described above, which yield new insights into our understanding of network formation. The basic impact of limited observation on social networks is largely independent of the network setting (mutual consent or non-cooperative) in that it always leads to an increased set of stable networks. Moreover, given that standard refinements lose power as observation becomes more limited, this paper suggests that the effects of limited observation are not so easily overcome. That said, this can actually be a good thing in settings in which efficient networks might only be stable under limited observation. I conclude the paper by briefly discussing how future research can build on these contributions.

2 Network Basics

Consider a set of players $N = \{1, ..., n\}$. In what follows, let each player represent a node in a *graph* or *network* g (I use "graph" and "network" interchangeably.) Let each arc (i.e., link or tie) in the g represent a binary or pairwise relation between the respective two nodes.

Let G be the set of all possible networks, i.e., the set of all possible combinations of binary relations. Let $ij \in g$ denote that there is a tie between i and j in g, while $ij \notin g$ implies no direct tie. Because mutual consent networks are studied here, these ties are *non-directed* in that $ij \in g \Leftrightarrow ji \in g$.

Let $g + ij$ represent the graph resulting from the addition of a link between i and j to g, while $g - ij$ represents the graph g if the ij link is removed.

Say that there is a *path* between i and j if either $ij \in g$ or if there exists m players $i_1, i_2, ..., i_m$, distinct from one another, such that $\{ii_1, i_1i_2, i_2i_3, ..., i_mj\} \subset g$. Let $d_g(i, j)$ be the *distance* in ties between i and j along the shortest path between i and j in graph g. For convenience, let $d_g(i, j) = \infty$ if $ij \notin g$, and let $d_g(i, i) = 0$.

Let $N_i(g) \subseteq N$ be the set of players in i's *component*; that is, the set of players that each have a path to i: $N_i(g) = \{j \in N | d_g(ij) < \infty\}$. Let $L_i = \{j \in N | ij \in g\}$ be the set of players with whom i has a direct link.

Let $v(g)$, $v : G \to \mathbf{R}$ be the *value* function that assigns a real number value to each graph, and let V be the set of all value functions. As will be seen in the model studied later, the v function can be thought of as a state variable will determine, in part, each player's utility. Let $Y_i : G \times V \to \mathbf{R}$ be i's utility function.

I follow Jackson and Wolinsky (1996) and say that g is *efficient* given v if $v(g) \geq v(g')$ for all $g' \in G$. It is often the case (as in the model below) that $v(g) = \sum_{i \in N} Y_i(g, v)$, so that efficiency would then represent the sum of utilities.[2]

3 Conjectural Pairwise Stability

Many social networks of interest require the mutual consent of both parties for a bilateral tie to exist. Applying non-cooperative equilibrium concepts to study mutual consent network is not possible without additional assumptions about link proposals and acceptances. In acknowledgment of this fact, Jackson and Wolinsky (1996) introduced the notion of *Pairwise Stability (PS)* as a coalitional stability concept for networks which considers only two-player coalitional deviations.

> **Definition 1:** Graph $g \in G$ is *Pairwise Stable* with respect to $v \in V$ and $\{Y_i\}_{i \in N}$ if
>
> (i) for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and
>
> (ii) for all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$.

Condition (i) requires that each player of an existing tie is better off with the tie than without it. Under condition (ii), if one player strictly prefers a new tie and the other is indifferent, then that tie will be formed.

Jackson (2003) acknowledges that the advantage of the PS concept is its ease of use in application, and that its primary limitation is that it allows for only bilateral deviations.[3] Nevertheless, PS turns out to be a strong enough notion of stability for many networks of interest, and although I revisit this issue of multi-player deviations in Section 5, this paper's

[2] A Pareto efficiency concept may be considered more appropriate in networks without side payments like that studied herein, but I follow Jackson and Wolinsky's usage for easier comparison.

[3] Dutta and Mutuswami (1997) allow for coalitions of more than two players, and Bienenstock and Bonacich (1997) discuss using the core concept. Goyal and Vega-Redondo (2005) use a different refinement that allows a pair of players to simultaneously remove and add links.

main concern is with a different aspect of the concept. While PS is too weak in allowing for only single bilateral deviations, it is also too strong in requiring individuals to have correct beliefs about the current state of both g and v.

I define a generalization of PS that allows individuals to have incorrect beliefs about g and v. In essence, I combine PS with existing Conjectural Equilibrium concepts. A Conjectural Equilibrium is one where each player chooses a best response to her beliefs about the other players' actions, and where her possibly incorrect beliefs are not contradicted by her available information about others' actions.[4] McBride (2005) generalized this concept to further allow for incomplete information (he calls his concept Generalized Conjectural Equilibrium).

To consider incorrect conjectures requires additional notation. Formally, let $\pi_i \in \triangle(G \times V)$ be a probability distribution over the possible states of the world (g, v). Call π_i i's beliefs. Also, let $m_i : G \times V \to M_i$, be i's *message* or *signal* function such that each state of the world yields a message m_i in message space M_i.

As defined, the full information settings is a message function profile such that for each i, $m_i(g, v) \neq m_i(g', v')$ for any distinct $(g, v), (g', v') \in \{G \times V\}$, i.e., each state of the world gives a unique message so that the state is perfectly distinguishable. In general, however, it is possible that two or more states of the world give the same signal to i.

Following the literature [Gilli (1999)], I place two initial restrictions on m_i.

Assumption 1: If $m_i(g, v) = m_i(g', v')$ then $Y_i(g, v) = Y_i(g', v')$.

Assumption 2: Fix $v, v' \in V$.

(i) If $ij \in g$ and $m_i(g, v) = m_i(g', v')$ then $ij \in g'$, and

(ii) If $ij \notin g$ and $m_i(g, v) = m_i(g', v')$ then $ij \notin g'$.

The idea behind Assumption 1 is that each player (e.g., after some network formation process) should learn her realized payoff, and so her set of information in an equilibrium should include information about that payoff. Assumption 2 says that i must be aware of

[4] For an extended discussion of the Conjectural Equilibrium concept, see Battigalli, Gilli, and Molinari (1992).

her own decision to give consent or not give consent to each possible bilateral tie, and this must be reflected in her signal.

We can now define a *Conjectural Pairwise Stable (CPS)* network.

Definition 2: Graph $g \in G$ is *Conjectural Pairwise Stable (CPS)* with respect to value function $v \in V$, utility functions $\{Y_i\}_{i \in N}$, message functions $\{m_i\}_{i \in N}$, and beliefs $\{\pi_i\}_{i \in I}$ if

(i) for all $ij \in g$, $Y_i(g,v) \geq \sum_{(g',v') \in G \times V} \pi_i(g',v') Y_i(g'-ij,v')$ and $Y_j(g,v) \geq \sum_{(g',v') \in G \times V} \pi_j(g',v') Y_j(g'-ij,v')$,

(ii) for all $ij \notin g$, if $Y_i(g,v) < \sum_{(g',v') \in G \times V} \pi_i(g',v') Y_i(g'+ij,v')$ then $Y_j(g,v) > \sum_{(g',v') \in G \times V} \pi_j(g',v') Y_j(g'+ij,v')$, and

(iii) for each i, $m_i(g',v') = m_i(g,v)$ for any $(g',v') \in G \times V$ s.t. $\pi_i(g',v') > 0$.

In words, a network is CPS if (i) each side of each existing bilateral tie believes she is better off keeping the tie, (ii) at least one party of any additional tie believes she is strictly worse off by forming the new tie, and (iii) no player's beliefs are contradicted by her signal.

Holding the value, utility, and message functions fixed, it is often possible that two (or more) different beliefs profiles $\{\pi_i\}_{i \in N}$ can be combined with g to make a CPS. Thus, it will often be useful to consider the set of CPS networks as the set of all $g \in G$ for which there exist beliefs that sustain g as a CPS. With this in mind, an issue of primary interest is how the set of CPS networks under one profile of message functions compares with the set of CPS networks under another message profile. In general, we cannot say anything without knowing the specific relationship of those two message function profiles. One special instance for which we can say something concrete is when one message function profile yields at least as much information as another profile.

Definition 3: Fix v and $\{Y_i\}_{i \in N}$. Say that message function profile $\{m_i\}_{i \in N}$ has *monotonically more information* than $\{m_i'\}_{i \in N}$, denoted $\{m_i\}_{i \in N} > \{m_i'\}_{i \in N}$, if

(i) $m_i'(g,v) \neq m_i'(g',v') \Rightarrow m_i(g,v) \neq m_i(g',v')$ for any $(g,v),(g',v') \in \{G \times V\}$ for all i, and

(ii) there exists $(g,v),(g',v') \in \{G \times V\}$ such that $m_i'(g,v) = m_i'(g',v')$ and $m_i(g,v) \neq m_i(g',v')$ for at least one i.

Condition (i) says that if the m' message functions yield different signals under two states, then under m, which is more informative, they must also yield different signals. Condition (ii) states that there must also exist two states that are distinguishable under m but not under m'. Of course, we can similarly define a message function profile that has *monotonically less information* than another profile.

Remark 1 immediately follows.

> **Remark 1:** *Fix v and $\{Y_i\}_{i\in N}$, and consider $\{m_i\}_{i\in N}$ and $\{m_i'\}_{i\in N}$ such that $\{m_i\}_{i\in N} > \{m_i'\}_{i\in N}$. Then any CPS g under $\{m_i\}_{i\in N}$ is also a CPS g under $\{m_i'\}_{i\in N}$, but the converse is not necessarily true.*

The logic behind Remark 1 is straightforward. Changing message functions while holding the value and utility functions fixed only changes the restrictions placed on beliefs in a CPS. Any restriction placed on beliefs under $\{m_i'\}_{i\in N}$ will also restrict beliefs under $\{m_i\}_{i\in N}$, but $\{m_i\}_{i\in N}$, because it contains more information, will place additional restrictions on beliefs not placed on beliefs under $\{m_i'\}_{i\in N}$. Thus, if the network meets the stricter requirements for CPS under $\{m_i\}_{i\in N}$, it will certainly meet the requirements for CPS under $\{m_i'\}_{i\in N}$, but if a network meets the CPS conditions under $\{m_i'\}_{i\in N}$ it might not meet the CPS conditions under $\{m_i\}_{i\in N}$. A further implication of Remark 1 is that any PS network is a CPS network under any message function profile. The full information message function profile provides monotonically more information than any other profile.

One last note deserves mention. Like the Conjectural Equilibrium concept, the CPS concept does not place any restrictions on beliefs other than that they are not contradicted by available information. The lack of additional restrictions matches, in spirit, the motivation for this paper. If individuals have very limited information about their network, it is not immediately obvious what additional restrictions should be placed on their beliefs. However, the lack of restrictions does potentially allow for a large set of stable networks. I return to this issue in Section 5.

4 The Connections Model

In this section, I use the CPS concept to study the relationship between network observation and network stability in a version of Jackson and Wolinsky's (1996) connections model. After presenting the basic model and describing features of PS networks, I address the issue of limited observation, and then present the main analytical results. In my analysis, I will not fully characterize the set of stable networks, but will instead identify particular properties of the set of stable networks under different levels of observation. My goal is to make substantive—but not necessarily exhaustive—comments about how the set of stable networks changes as observation changes.

4.1 The Basic Model

Suppose links represent social relationships that yield benefits such as favors, information, emotional support, etc., and the value of these benefits will depend on the characteristics of the person linked to. Formally, let $N = \{1, ..., n\}$ be the set of individuals, and let $v_i \in (0, \infty)$ represent the inherent value of individual i to all others. Ties are the only way to access others' valuable benefits, yet forming a tie is costly in that each side of the tie pays cost $c \in (0, 1]$. This cost represents the investment, opportunity cost, etc., of forming and maintaining the link.

An individual receives benefits from her direct ties, but she also receives benefits from indirect relationships, although the benefits of an indirect relationships diminish in the distance of the relationship. This is reflected in the following payoff function:

$$Y_i(g, v) = \sum_{j \neq i} \delta^{d_g(i,j)} v_j - \sum_{j \text{ s.t. } ij \in g} c,$$

where δ, $0 < \delta < 1$, measures the *decay* in benefits.

For notational convenience, let $v = (v_i)_{i \in N}$ be the profile of players' types, and let V be the set of such profiles. Thus, the state of the world is characterized by a (g, v) combination.

Figure 1(a) illustrates one possible network with $n = 5$. There is one multi-player component consisting of everyone but player 3, who is isolated. Players 1 and 2 have two direct links, player 4 has one direct link, and player 5 has three direct links. If $v = (1, 1, 2, 2, 2)$, $\delta = \frac{1}{2}$, and $c = 1$, then we can calculate 1, 3, and 5's payoffs as follows:

$$\begin{aligned} u_1 &= \delta (v_2 + v_5) + \delta^2 (v_4) - |L_1(g)| c = \frac{1}{2}(2+2) + \left(\frac{1}{2}\right)^2 (2) - 2(1) = \frac{1}{2} \\ u_3 &= 0 \\ u_5 &= \delta (v_1 + v_2 + v_4) - |L_5| c = \frac{1}{2}(1+1+2) - 3(1) = -1. \end{aligned}$$

Note that an individual, such as player 1, can receive benefits from both her direct links and any indirect links, although the benefits are decreasing in distance due to decay. It may also be better to remain isolated than to be in a component, e.g., isolated player 3 is better off than player 5.

There are many reasons to study this model. First, it captures generic elements of many actual social networks: individuals benefit from direct ties; individuals also benefit from indirect ties, although the benefits do diminish in distance; and individuals must exert a cost to maintain relationships. Second, a symmetric version of this model has already been examined by Jackson and Wolinsky (1996). They consider the case of homogenous types $v_i = 1$ for all i, perfect monitoring of links, and complete information of types. My model is a natural generalizes the setting to include heterogeneous types.[5] Finally, allowing for heterogenous types is necessary to make an examination of incomplete information non-trivial.

4.2 PS Networks

Before turning to limited observation, I first examine PS networks (i.e., the set of CPS networks under full information). Some further terminology and notation will prove useful.

[5] Johnson and Gilles (2000) and Jackson and Rogers (2005) also consider heterogeneous types in the connections model, yet they do not consider limited observation.

If $v_i > \frac{c}{(\delta-\delta^2)}$, then say that i is *very-high-valued*, and denote it type category VH. If $\frac{c}{\delta} < v_i < \frac{c}{(\delta-\delta^2)}$, then say that i is *high-valued*, H. If $v_i < \frac{c}{\delta}$, then say that i is low-valued, L. Summarizing:

$$L < \frac{c}{\delta} < H < \frac{c}{(\delta - \delta^2)} < VH.$$

These type categories have useful properties. If i is linked to j and j is low-valued with no other links, then i is better off removing her link with j. Conversely, if j were high-valued or very-high-valued and with no other links, then i is better off maintaining that link. Finally, if there is a path between i and j, j is very-high-valued, but i and j are not directly connected, then, as will be shown in Proposition 1(i), i is still better off being directly linked to j.

Also, let Y^{ij} be the marginal benefit of link ij to i holding g fixed, so that $Y^{ij} = Y_i(g+ij,v) - Y_i(g,v)$ if $ij \notin g$ and $Y^{ij} = Y_i(g,v) - Y_i(g-ij,v)$ if $ij \in g$.

I can now describe some properties of PS networks.

Proposition 1: *Fix $v \in V$, and consider full information.*

(i) For any i,j that are very-high-valued, then any PS g must have $ij \in g$.

(ii) If all players are very-high-valued, then the unique PS network is the complete graph.

(iii) Any PS network has at most one multi-player component in which any isolated individual is low-valued.

(iv) Any low-valued individual in a PS network cannot be a stem with $|L_i| = 1$.

(v) The empty network is PS only if $n-1$ or more individuals are low-valued.

Proof: (i) Suppose the contrary, i.e., there exists PS g with $ij \notin g$ such that $(\delta - \delta^2)\, v_i > c$ and $(\delta - \delta^2)\, v_j > c$. A pairwise deviation that adds ij to g yields i the smallest marginal increase in utility when i already receives j's benefits through another link such that j is two links away. In this case, the marginal benefit to i of the direct link is $(\delta - \delta^2) v_j$, and this is greater than c by assumption. Since j's marginal benefit of the ij link are identical, both want to make the link, which violates PS.

(ii) Follows from (i).

(iii)[6] Consider pairwise stable g with two multi-player components N_i, which includes ij, and N_k, which includes kl. Since g is pairwise stable, it must be that $Y^{ij} \geq 0$. Since i and j are in the same component, it follows that $Y^{kj} > Y^{ij} \geq 0$, since k will also receive at least $\delta^2 v_i > 0$ from the indirect link to i which is not included in Y^{ij}. By similar logic, $Y^{jk} > Y^{lk} \geq 0$. This contradicts pairwise stability, since both j and k are strictly better off forming a link. That any isolated player must be low valued follows directly.

(iv) Follows directly since any i is strictly worse linking to an isolated player j with $v_j < \frac{c}{\delta}$.

(v) Follows from (iv). □

Figure 2 illustrates various parts of Proposition 1. Figures 2(a)-(c) focus on H or VH players in PS networks. Figure 2(a) is not PS since the top left H and the bottom left H are both strictly better off linking to each other (Proposition 1(i)). Figure 2(b) is also not PS since it has two VH players not directly connected. Figure 2(c), however, is PS since all VH players are directly connected and H is not isolated. Figures 2(d) illustrates a network that violates Proposition 1(iii). Any multi-player component must have total value greater than or equal to an H player, and so having multiple multi-player components is akin to having multiple H or VH players, i.e., each cannot be isolated in a PS network if there are more than one of them. The top H and top VH are both better off linking to each other. Figure 2(e) is not PS since L is a stem (Proposition 1(iv)). Finally, Figure 2(f), an empty network, is PS because the Ls will not link to the other Ls, and even though they want to link to H, H will not link to them (Proposition 1(v)).

[6] This proof uses logic from the proof of Jackson and Wolinsky's (1996) Proposition 2(i).

As I show below, each condition described here for PS networks can be violated in CPS networks when there is limited network observation. However, we must first specify the observational environment.

4.3 x/y-link Observation

Applying the CPS concept to the connections model requires an explicit characterization of players' message functions. I will use the formal notion of x/y-link observation introduced by McBride (2005) that depicts imperfect monitoring and incomplete information in social networks.[7] Given some (g, v) combination with x/y-link observation, i observes all links that are within distance x links, and she observes v_j for all j who are within distance y links. I will initially assume $x \geq 1$ and $0 \leq y \leq x$. These are natural restrictions. An individual should know with whom she has direct links at the least ($x \geq 1$). She should only be able to observe j's type if she observes j interact in the network (y cannot exceed x). At the same, observing j's links does not necessarily imply that j's type can be observed (x can exceed y).

Figure 3(a) illustrates what player 1 observes in the least observation setting of $x = 1$ and $y = 0$. The links within the dotted boundary are observed by 1, and the line under a player denotes that 1 observes that player's type. Player 1 observes only her own direct links and her own type. Figure 1(b) illustrates what 1 observes with $x = 1$ and $y = 1$, and Figure 1(c) illustrates what 1 observes with $x = 2$ and $y = 1$. Increasing either x or y or both leads to monotonically more information, but so long as x and y are both finite, a player can never observe the links or type of a player in another component. Moreover, it will often be the case that two players will observe different parts of the network. Figure 1(d) illustrates what player 2 observes with 1/0-link observation, which differs from what 1 observes under 1/0-link observation.

I use the x/y-link observation signal structure for various reasons. First, x/y-link observation, by design, mimics to some extent the observation present in actual networks.

[7] McBride's (2004) x/y-link observation generalizes the x-link observation developed in McBride (forthcoming) for games with only imperfect monitoring (and complete information).

Empirical work by sociologists finds that individuals have limited "horizons of observability" in that they are more likely to correctly perceive others' ties or personal characteristics if they are closer in the network [Laumann (1969), Friedkin (1983), Kumbasar, Romney, and Batchelder (1994), Bondonio (1998), Casciaro (1998)]. Second, the x/y-link observation notion has already been used by McBride (2005) in the study of non-cooperative network formation under limited observation. In fact, it is the only existing formal game theoretic signal structure for network games with limited network observation. Using it here thus allows me to compare my results with those of the earlier research.

Finally, I note that I can use this concept even though it was first presented in the context of a non-cooperative network model. The concept defines signals as a function of the graph and value function and not the strategic nature of the link formation process.

4.4 CPS Networks

We now turn to an examination of CPS networks, and one final new notation will be used in our analysis. Define $Y_{\pi_i}^{ij}$ to be the expected benefits that go to i from the ij link given i's beliefs π_i in (g, v). Formally,

$$Y_{\pi_i}^{ij} = \begin{cases} \sum_{(g',v') \in G \times V} \pi_i(g', v') \left(Y_i(g' + ij, v') - Y_i(g, v)\right) & \text{if } ij \notin g \\ \sum_{(g',v') \in G \times V} \pi_i(g', v') \left(Y_i(g, v) - Y_i(g' - ij, v')\right) & \text{if } ij \in g \end{cases}.$$

This is the expectation version of Y^{ij} defined in Section 4.2, so that Y^{ij} and $Y_{\pi_i}^{ij}$ are equivalent if π_i assigns probability 1 to the true state g.

The next result examines the least information setting of $x = 1$ and $y = 0$. Surprisingly, any network that makes each player no worse off than being isolated will be a CPS network.

Proposition 2: *Fix $v \in V$ and suppose 1/0-link observation. Then g is a CPS network if and only if $Y_i(g, v) \geq 0$ for all i.*

Proof: (Necessity) Consider g that is a CPS network with associated beliefs profile $\{\pi_i\}_{i \in N}$ and with $Y_i(g, v) < 0$ for some i. From the definition of $Y_{\pi_i}^{ij}$ for

$j \in L_i$:

$$
\begin{aligned}
Y_{\pi_i}^{ij} &= \sum_{(g',v')\in G\times V} \pi_i(g',v')\left(Y_i(g,v) - Y_i(g'-ij,v')\right) \text{ if } ij \in g \\
Y_{\pi_i}^{ij} &= \sum_{(g',v')\in G\times V} \pi_i(g',v') Y_i(g,v) - \sum_{(g',v')\in G\times V} \pi\left({}_i g',v'\right) Y_i(g'-ij,v') \\
Y_{\pi_i}^{ij} &= Y_i(g,v) - \sum_{(g',v')\in G\times V} \pi_i(g',v') Y_i(g'-ij,v') \quad \text{by CPS condition (iii).}
\end{aligned}
$$

By CPS condition (i), the first term on the RHS is weakly greater than the second term on the RHS. This implies, first, that the LHS is weakly positive: $Y_{\pi_i}^{ij} \geq 0$ for all $j \in L_i$. It also implies that the LHS is weakly less than the first term on the RHS: $Y_{\pi_i}^{ij} \leq Y_i(g,v) \in L_i$ for all j. Putting these conditions together, we get a contradiction: $0 \leq Y_{\pi_i}^{ij} \leq Y_i(g,v) < 0$.

(Sufficiency) Consider g in which $Y_i(g,v) \geq 0$ for all i. I now construct π_i for each i to sustain g as CPS. First suppose $|L_i| \geq 1$. Consider a network g' which has ij for each $j \in L_i$ and no others links, e.g., i is the center of a star network. Consider v' as follows: $v_i' = v_i$; $v_j' = \frac{Y_i(g,v)+[L_i]c}{\delta|L_i|} = \frac{Y_i(g,v)}{\delta|L_i|} + \frac{c}{\delta}$ for all $j \in L_i$, which means that all of i's payoff is spread evenly among all $j \in L_i$; and $v_j' = \varepsilon > 0$ small for all $j \notin L_i$, $j \neq i$. Since $Y_i(g,v) \geq 0$, it follows that $v_j' \geq \frac{c}{\delta}$, so each v_j' is high-valued. For each i with $|L_i| \geq 1$, respectively, choose π_i such that probability 1 is assigned to (g',v'). With these beliefs π_i for i, she will want to remain the center of the star g', and this is true for all i. Thus, these beliefs, combined with g, meet the CPS conditions (i)-(iii) for each i.

Now consider i with $L_i = \emptyset$. Consider v' such that $v_i' = v_i$ and $v_j' = \varepsilon > 0$ small for all $j \neq i$. For any i with $L_i = \emptyset$, choose π_i such that probability 1 is assigned to (g',v'), where g' is the empty network with no links. These beliefs, combined with g, meet the CPS conditions for i. $\square$

To state this result precisely: with 1/0-link observation, for any network g in which each player's payoff is weakly greater than being isolated, there exists a profile of beliefs for each i that will sustain g as a CPS network. The $Y_i(g,v) \geq 0$ condition is naturally interpreted as a participation constraint for i, so the set of CPS networks is extremely large. Unlike the set of (full information) PS networks, the empty network is always CPS under finite x and y no matter the v, while the empty network is PS only if at least $n-1$ players are low-valued. Even if multiple players are high-valued, an isolated player under 1/0-link observation can incorrectly believe that everyone else is low-valued and isolated, and thus not want to form any links. Remaining isolated is i's best response given these beliefs, and she receives no information to contradict her beliefs. Such incorrect beliefs are not allowed in a PS network since v is known. For similar reasons, CPS networks can have multiple multi-player components since a player in one component may believe anyone not in her component is low-valued and isolated, and high-valued and very-high-valued players may miss opportunities to link to one another.

In fact, if all players are high-valued ($v_i > \frac{c}{\delta}$), then the set of CPS networks is the set of all possible graphs G because any g will meet i's participation constraint. This "anything goes" result demonstrates in an extreme manner the role that observation plays in network stability.

A few examples will help illustrate these findings. The left-most network in Figure 4(a) with an isolated H player can be sustained as CPS with the beliefs depicted in the center and right networks: H believes that all others are low-valued and isolated (center), and the VHs believe that the isolated H player is low-valued. Given these beliefs, each player is doing her best action, and no player observed any information that contradicts her beliefs. Figure 4(b) is a network that has two multi-player components, and it is CPS if members of one component think their component is the only component and that all others are low-valued and isolated. The lack of payoff increasing connections can be more extreme as in Figure 4(c), since the empty network can always be sustained as a CPS. Finally, Figure 4(d) depicts a network with a low-valued stem that can be sustained as a CPS when the player linked to that stem believes that the stem is not low-valued.

The large set of CPS networks under 1/0-link observation includes networks that are less efficient than those in the set of the PS networks. If two players are very-high-valued, then it is both individually and socially optimal for them to be directly linked in a PS network, and they will be directly linked in any PS network. This connection is not guaranteed with limited observation. The disconnected network in Figure 4(a) is CPS, but the unique PS network for this profile of player types is the connected network in Figure 2(c). Thus, severe observational limitations will generally lead to the existence CPS networks which yields lower social utility that PS networks since players may not observe payoff improving connections.

On the other hand, it might be possible that there is a CPS network that yields a higher sum of utilities than does the PS network with the highest sum of utilities. Figure 4(d), for example, is a CPS network under 1/0-link observation, but it is not PS since there is a low-valued stem. The unique PS network would have the three VHs directly connected and L isolated, but note that if the VHs have very large values, then the network in Figure 4(d) would generate higher social utility than the PS network. In this case, limited observation leads players to miss payoff improving link removals.

That less efficient CPS networks arise as observation decreases matches the same finding reported by McBride (forthcoming, 2005) in the context of a non-cooperative communication network model. That more efficient CPS networks arise as observation decreases is a new finding to the game theoretic literature on social networks. In retrospect, both findings make intuitive sense. Decreasing observation allows players to maintain incorrect beliefs, and this means that individuals can choose an action that is individually detrimental but not known to be so. This individually detrimental action can be socially detrimental, as in Figure 4(a), or socially beneficial, as in Figure 4(d).

Since increasing x or y leads to monotonically more information for each player, we know from Remark 1 that increasing x or y will eliminate many, but not all, of the equilibria that exist under 1/0-link observation. Proposition 3 illustrates this fact by listing some features of the set of CPS networks for generic (finite) x and y.

Proposition 3: *Fix $v \in V$ and suppose x/y-link observation with $x < \infty$ and $y < \infty$.*

(i) If $x > 1$ and $y \geq 1$, then any CPS g must have no low-valued stems.

(ii) Consider a CPS g in which there exists very-high-valued i and j such

that $d_g(i,j) \leq y$. *Then* $ij \in g$.

(iii) If there are at least four high-valued players, then the set of CPS networks includes networks with multiple multi-player components.

(iv) The empty network is always CPS.

Proof: (i) Suppose the contrary, that CPS g has a low-valued stem j, i.e., $\delta v_j < c$, and ij is j's only link. Because $x > 1$ and $y \geq 1$, i must observe that j is low-valued and a stem, and this must be reflected in her CPS beliefs π_i. But given these beliefs, keeping her link with j is not a best response, which contradicts CPS.

(ii) Suppose the contrary, that CPS g has $i, j \in N_i$ s.t. $c < \left(\delta - \delta^2\right) v_i$, $c < \left(\delta - \delta^2\right) v_j$, and $d(i,j) \leq y$, but where $ij \notin g$. Since $d(i,j) \leq y$, i and j both see each other's type, so each's beliefs must assign the correct type to the other. Each thus believes she is strictly better off by linking to the other, which contradicts CPS.

(iii) Pick v such that i, j, k, and l are high-valued, and any other player $m \neq i,j,k,l$ can be either low- or high-valued. Construct g such that it contains ij and kl and no other links. Let π_i and π_j each assign probability 1 to (g', v') where their link is the only link in g', and where v' has $v'_i = v_i$, $v'_j = v_j$, and $v'_m = \overline{v} < \frac{c}{\delta}$ for $m \neq i,j$. Let π_k and π_l each assign probability 1 to (g'', v'') where their link is the only link in g'', and where v' has $v'_k = v_k$, $v'_l = v_l$, and $v'_m = \overline{v}$ for $m \neq k,l$. Finally, let π_m, for all $m \neq i,j,k,l$ assign probability 1 to the empty network and all others having value $\overline{v}$. No player's beliefs are contradicted by their observation, and each player's links are individually optimal. Thus, g is CPS with multiple non-empty networks.

(iv) Let i believe that assign probability 1 to a state in which all other $j \neq i$ be isolated and have $v_j < \frac{c}{\delta}$. Because i is isolated and x and y are finite, i receives no information to contradict her beliefs. Given these beliefs, i believes she is better off not forming any links. The CPS conditions are met for i. Construct similar beliefs for all i to complete the CPS. $\square$

Propositions 3(i)-(ii) show how increasing observation can sometime hurt while at other times help efficiency. With sufficiently high observation, low-valued stems are observed and cannot exist in a CPS even if they are socially optimal. Part (i) shows that slightly increasing observation above 1/0-link observation is sufficient for this to happen. Part (ii) shows that increasing observation also allows very high-valued players to identify other very-high-valued players in their component, thus leading them to make direct connections. Of course, CPS networks can have very high-valued players not directly linked if the distance between them is sufficiently large.

Parts (iii) and (iv) illustrate more general features of CPS networks when players have less than full information. For multiple multi-player components to exist in a CPS network, it is necessary that the benefits to each player in each component must exceed the costs of being in the component. Under full information, a player in each component knows she is better off linking to the player in the other component, so multiple multi-player components cannot exist in a PS network. When players do not observe other components, however, they will not know of those beneficial link formations. Having four high-valued players is sufficient for multiple multi-player components to exist but not necessary. Wheel components made of four or more low-valued players whose values are lower than but close to $\frac{c}{\delta}$ can also be CPS. Finally, for similar reasons, the empty network is always CPS.

These findings add a new insight into the relationship between stability and efficiency. Jackson and Wolinsky (1996) used the connections model to demonstrate how many efficient networks are not stable. My results suggest that the tension between stability and efficiency is potentially tempered by individuals' observation, although not necessarily so. Since reducing observation will generally lead to the existence of less efficient stable networks, observational limitations can hurt efficiency. This will necessarily happen if the efficient network is PS. However, in some settings where PS networks are inefficient, reducing observation can lead to the existence of CPS networks that yield higher social utility.

Whether observation helps or hurts efficiency thus depends on the particular profile of types v. For example, when all individuals are very-high-valued, then the unique PS network is the complete graph. Because this graph is also the unique efficient graph, any CPS graph under limited observation (other than the complete graph) can only be inefficient. As shown in Figure 4(d), however, the presence of low-valued players can yield a limited observation with higher sum of utilities than the PS network.

5 Refinements of CPS

5.1 Rationalizability

Rubinstein and Wolinsky (1994) and Gilli (1999) acknowledge that the Conjectural Equilibrium concept for non-cooperative games does not require players' conjectures to be justified, and instead only requires that their conjectures not be contradicted by their signals. They consider imposing common knowledge of rationality as a way to refine players' beliefs so that each player's beliefs must reflect optimal play on the part of the other players. We can impose common knowledge of rationality by assuming players commonly know x and y, that everyone has x/y-link observation, and that everyone plays a best response to her conjectures. Individual i must justify her beliefs about j's beliefs and actions given her beliefs about j's signal, and j must in turn rationalize her beliefs about k's actions and beliefs given her beliefs about k's signal, and so on.

Whether this additional restriction on beliefs will refine the set of CPS in the connections model will depend on how informative are the players' signals.

Proposition 4: *Fix $v \in V$.*

(i) Suppose 1/0-link observation, and consider a CPS g such that $v_i \geq \frac{c}{\delta}$ for all i with $|L_i| = 1$ and $Y_i(g, v) \geq \left(\frac{c}{\delta} - v_i\right) \frac{|L_i|}{|L_i|-1} - c$ for all i with $|L_i| > 1$. Then g is also rationalizable CPS.

(ii) Suppose $x = \infty$ (perfect monitoring) and $y < \infty$. Then there can exist g that are CPS but not rationalizable CPS.

(iii) Suppose x/y-link observation with $x < \infty$ and $y < \infty$. Then the

empty network is always CPS and rationalizable CPS.

Proof: (i) Consider (g, v) in which $Y_i(g, v) \geq \left(\frac{c}{\delta} - v_i\right) \frac{|L_i|}{|L_i|-1} - c$ for all i with $|L_i| > 1$. First consider isolated i in g. Let $\pi_i(g', v') = 1$ s.t. g' is empty, $v'_i = v_i$, and $v'_j = \varepsilon < \frac{c}{\delta}$ for all $j \neq i$. The CPS conditions are met for i with 1/0-link observation. These beliefs about (g', v') can be rationalized as follows. Let i believe that $\pi_j(g'', v'') = 1$ where g' is empty and $v''_k = \varepsilon$ for all k. Given this belief by i about j's beliefs, j's action is optimal, and his beliefs are not contradicted by available evidence in (g', v'). In a similar manner, j rationalize her beliefs about (g'', v''), and so on. We have thus constructed an infinite chain of justification starting from isolated i.

Now consider non-isolated i with $|L_i| \geq 1$. In proving Proposition 2, I showed that the CPS conditions were met for i with $|L_i| \geq 1$ if π_i assigned probability 1 to the state (g', v') in which i was the center of a star network that consisted only of her links, where each $j \neq i$ in the star had value $v'_j = \frac{Y_i(g,v)}{\delta |L_i|} + \frac{c}{\delta}$, and where all others were low-valued and isolated. I now show in 3 steps that this belief yields an infinite chain of justification.

1. *i can rationalize k who is isolated in g'.* Consider k with $|L_k| = 0$ in g'. Because g' is a star with center i, these k's are all those not in L_i. For any player k who is isolated in g', i can rationalize k's actions and beliefs as done in the first paragraph of this proof. Thus, i rationalizes k's beliefs and actions in (g', v').

2. *i can rationalize j who is linked in g'.* Consider j with $|L_j| \geq 1$ in g'. Because g' is a star, this includes all $j \in L_i$. Suppose π_j assigns probability 1 to state (g'', v'') with the following properties: let ij be the only link in g''; let $v''_j = v'_j = \frac{Y_i(g,v)}{\delta |L_1|} + \frac{c}{\delta}$, which implies that v'_j is high-valued or very-high-valued; let all other $k \neq j, i$ be low-valued and isolated; and let $v''_i = v_i + \delta(|L_i| - 1) v'_j$. (Note: this value v''_i, when multiplied by δ and then subtracting c, yields payoff $Y_j(g', v')$ to j.) As constructed, $Y_j(g', v') = Y_j(g'', v'')$ and $m_j(g', v') = m_j(g'', v'')$ under

1/0-link observation. Thus, CPS condition (iii) is met for j according to i. Moreover, j's action is a best response given π_j. CPS condition (ii) is met since she believes any additional link with make her strictly worse off. CPS condition (i) is met when she is better off remaining in g'' than removing her link to become isolated, which yields payoff 0:

$$\begin{aligned} Y_j\left(g'', v''\right) &\geq 0 \\ \delta\left(v_i + \delta\left(|L_i| - 1\right) v_j'\right) - c &\geq 0 \\ \delta\left(v_i + \delta\left(|L_i| - 1\right) \frac{Y_i\left(g, v\right) + c}{\delta\left|L_i\right|}\right) - c &\geq 0. \end{aligned}$$

If $|L_i| = 1$, then this condition becomes

$$\begin{aligned} \delta\left(v_i + \delta\left(1 - 1\right) \frac{Y_i\left(g, v\right) + c}{\delta}\right) - c &\geq 0 \\ \delta v_i &\geq \frac{c}{\delta}. \end{aligned}$$

Otherwise, if $|L_i| > 1$, then this condition becomes

$$Y_i\left(g, v\right) \geq \left(\frac{c}{\delta} - v_i\right) \frac{|L_i|}{|L_i| - 1} - c.$$

These two conditions are exactly those listed in Proposition 4(i). Thus, i can justify this belief and action for j.

3. *We construct an infinite chain of rationalization.* We now go one more level in the chain of justification by showing that k and j with the beliefs (from 1 and 2) can rationalize the belief. k can rationalize all others being isolated just as her own beliefs were rationalized. j's beliefs that j has only one link with i can also be rationalized. First, any isolated player can be rationalized as above. Second, j can believe i has beliefs π_i'' in which ij is the only link and in which v_j' is high-valued.. By the same logic above, this can be rationalized if condition (i) is met. Continuing the same logic over and over, we construct an infinite chain of justification.

(ii) An example will suffice. Suppose $n = 4$, $v = (\overline{v})$, $\overline{v} = \frac{c}{\delta} + \varepsilon$, $\varepsilon > 0$ small, for all i. Then g such that $12 \in g$ and $34 \in g$, but no other links in g is CPS but not rationalizable CPS. Let π_i for $i = 1, 2$ assign probability 1 to (g, v'), where $v' = \left(\frac{c}{\delta} + \varepsilon, \frac{c}{\delta} + \varepsilon, 0, 0\right)$. Similarly, let π_i for $i = 3, 4$ assign probability 1 to (g, v''), where $v' = \left(0, 0, \frac{c}{\delta} + \varepsilon, \frac{c}{\delta} + \varepsilon\right)$. With these beliefs, g is CPS.

These beliefs are not rationalizable CPS since 1 believes that 3 and 4 are maintaining a link that is not worth maintaining. In fact, no beliefs can sustain g as rationalizable CPS. With $x = \infty$, player 1 sees the other component. If 1 believes that 3 is rational, then 1 must believe 3 maintains that link because it makes 3 better off. But this means that for any rationalizable CPS beliefs for 1, 1 would also be better off linking with 4. By similar logic, 4 must believe she is better off linking with 1.

(iii) Follow from logic similar to that used in examining the isolated players in the proof of part (i). □

The key logic behind condition $Y_i(g, v) \geq \left(\frac{c}{\delta} - v_i\right) \frac{|L_i|}{|L_i|-1} - c$ in Proposition 4(i) is that each non-isolated player i must believe that she provides enough benefits to any neighbor j that i can rationalize j's choice to retain the ij link. The greater i's value, the lower i's payoff from sources other than j needs to be to satisfy the condition, while the lower i's value, the more i must get from other sources. In fact, if each i is high-valued (or very-high-valued) so that $v_i > \frac{c}{\delta}$, then the RHS of the condition is negative, which implies that the condition is trivially satisfied by any CPS network, which we know from Proposition 2 must have $Y_i(g, v) \geq 0$. It follows that if all players are high-valued, then every CPS network under 1/0-link observation is also rationalizable CPS. Hence, the "anything goes" result for CPS networks under 1/0-link observation arises even with the common knowledge of rationality refinement. In this case, the common knowledge of rationality restriction does nothing to refine the set of CPS networks.

While rationalizability does little or nothing under 1/0-link observation, more informative signals implies further restrictions on others' beliefs. This will, in turn, allow a player to further refine her rationalization of the other players, thereby refining the set of CPS networks. By increasing observation to perfect monitoring, for example, players observe other components. Attributing rational behavior to the members of the other component implies that their participation is justified by sufficiently high payoffs. This logic leads a member of each component to believe it must be worthwhile to link to that other component. Thus, players must believe in profitable pairwise deviation. Nonetheless, adding rationalizability cannot eliminate the empty network under finite x and y because isolated players can rationalize their beliefs that others are low-valued.

These findings reveal a more general result that the effect of adding common knowledge of rationality is greater when the signals are more informative. A related finding for non-cooperative games was made by Rubinstein and Wolinsky (1994), whereby common knowledge of rationality yields the set of rationalizable strategies when signals are perfectly uninformative (i.e., every state gives the same signal) but yields the set of Nash equilibria when signals are perfectly informative (i.e., each state gives a unique signal). In the connections model, increased informativeness of signals bring us closer to the set of PS networks. However, with finite x and y, the empty network is always rationalizable, so common knowledge of rationality does not bring us entirely back to the set of PS networks.

5.2 Coalitional Deviations

A limitation of the PS and CPS concepts is that only single link, or two-person, deviations are considered. In general, players may be allowed to deviate in coalitions of any size less than or equal to $m \leq n$.[8] If so allowed, there may be some networks that can withstand bilateral deviations but not deviations of more than two players. As with imposing common knowledge of rationality, the effect of allowing larger coalitions will be larger when players observe more of the network.

Proposition 5: *Fix $v \in V$.*

(i) Suppose $y = \infty$ (complete information). Then there can exist g that are CPS but not immune to large coalitional deviations.

[8] Dutta and Mutuswami (1997), for example, introduce the notion of *Strong Stability* which refines PS by allowing coalitions of any size. This concept does have problems with existence, however, since in some settings there may not be any networks that are impervious to any coalitional deviations.

(ii) Suppose x/y-link observation with $x < \infty$ and $y < \infty$. Then there exist beliefs such that the empty network survives coalitional deviations of any size.

Proof: (i) An example with $y = \infty$ illustrates. Suppose $n = 4$ and $v_i = \frac{c}{\delta} - \varepsilon$, $\varepsilon > 0$ small, for all i. Since all players are low-valued, the empty network which gives each player $Y_i = 0$ is CPS by Proposition 3(iv). However, a four-player deviation to a wheel network gives to each i a payoff

$$\begin{aligned} Y_i &= 2\delta\left(\frac{c}{\delta} - \varepsilon\right) + \delta^2\left(\frac{c}{\delta} - \varepsilon\right) - 2c \\ &= \delta c - 2\varepsilon\delta - \delta^2\varepsilon \end{aligned}$$

for all i, which is greater than 0 if ε sufficiently small. Since v is commonly known with $y = \infty$, this fact would be manifest in each i's beliefs, so each i would be willing to participate in such a deviation.

(ii) Consider π_i that assigns probability 1 to (g', v'), where g' is the empty network, where $v'_i = v_i$, and where $v'_j = \varepsilon < \frac{c}{\delta(n-1)}$. Notice that i cannot achieve a payoff equal to or greater than $\delta(n-1)\varepsilon - c < 0$, which is what she would get it in the infeasible case in which she received the largest possible benefits, $\delta(n-1)\varepsilon$, by making the fewest number of links, 1. Thus, there is no state g in which i is not isolated that gives i a payoff equal to or greater than that from being isolated. Hence, if i is isolated with beliefs π_i, there is no deviation that i would participate in. This logic applies for all i. □

In the case of larger coalitions, just as with adding common knowledge of rationality, the empty network again survives the refinement. The reason is that an isolated player with finite x and y receives no information about the others, and can maintain a belief that a deviation of any size will make her strictly worse off. Increasing observation, say by allowing complete information of types, does have refining power since there will be some CPS networks that do not survive multi-player deviations.

6 Conclusion

This paper examines mutual consent networks under limited observation. I introduce the CPS concept for mutual consent social networks with imperfect monitoring and incomplete information. I then use this concept to examine the role of limited observation in the stability and efficiency of networks in the connections model. While limited observation in general leads to the existence of inefficient networks that would not be PS under full information, it may also be true that if PS networks are inefficient, then limited observation may actually lead to the existence of more efficient CPS networks. I also report an "anything goes" result such that, under certain conditions on players' types, if observation is limited, then any network that meets each player's participation constraint is a CPS network. Finally, I show that neither the common knowledge of rationality refinement nor the large coalition deviation refinement is sufficient alone to eliminate many CPS networks. These restrictions lose refining power as observation becomes more limited.

Future research has many avenues to consider, two of which I mention here. First, theoretical research should examine how efficiency and observation interact in other strategic network formation games to determine the generality of the results presented here. The work to date, which includes this paper and McBride (forthcoming, 2005), has commonly found that decreasing observation leads to the existence of less efficient networks, and this finding will likely hold true in most any network setting. The finding here that more efficient stable networks might also arise is new to the literature, and whether it holds true in other settings is yet to be determined. Can general statements be made about the efficiency-observation relationship? For example, is there a class of positive externality games in which limited observation only hurts efficiency? Future work should look for such general statements.

A second line of research is to study how individuals act to overcome their limited observation. Since social networks have structured markets and other social interactions for centuries, individuals have likely developed mechanisms to overcome some of the inefficiencies that can persist due to limited observation. For example, what mechanisms have arisen to increase one's credibility about statements to a neighbor about one's own links? Since my results indicate that predicting what networks will form is nearly impossible when observation is very limited, a better understanding of the manner individuals overcome observational limitations may be necessary for making sharper inferences about network structure. McBride (2005) shows that observing others' links has different implications for network formation than does observing others' types. Are individuals more successful in observing others' types or actions? Answering this and the other questions will bring us closer to understanding how the social relationships that structure much of everyday social and economic activities form and evolve over time despite informational limitations.

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Figure 1

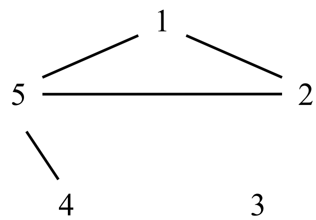
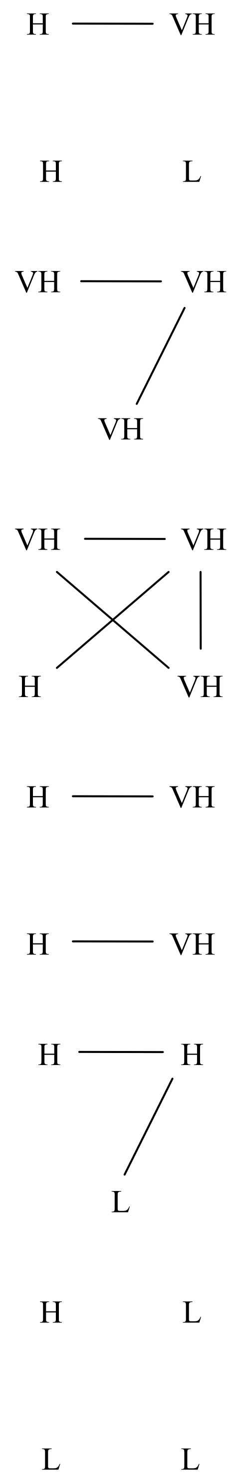


Figure 2



(a) Not PS. H is isolated.

(b) Not PS. Two VHs are not directly connected.

(c) PS.

(d) Not PS. Two multi-player components.

(e) Not PS. L is a stem.

(f) PS.

Figure 3

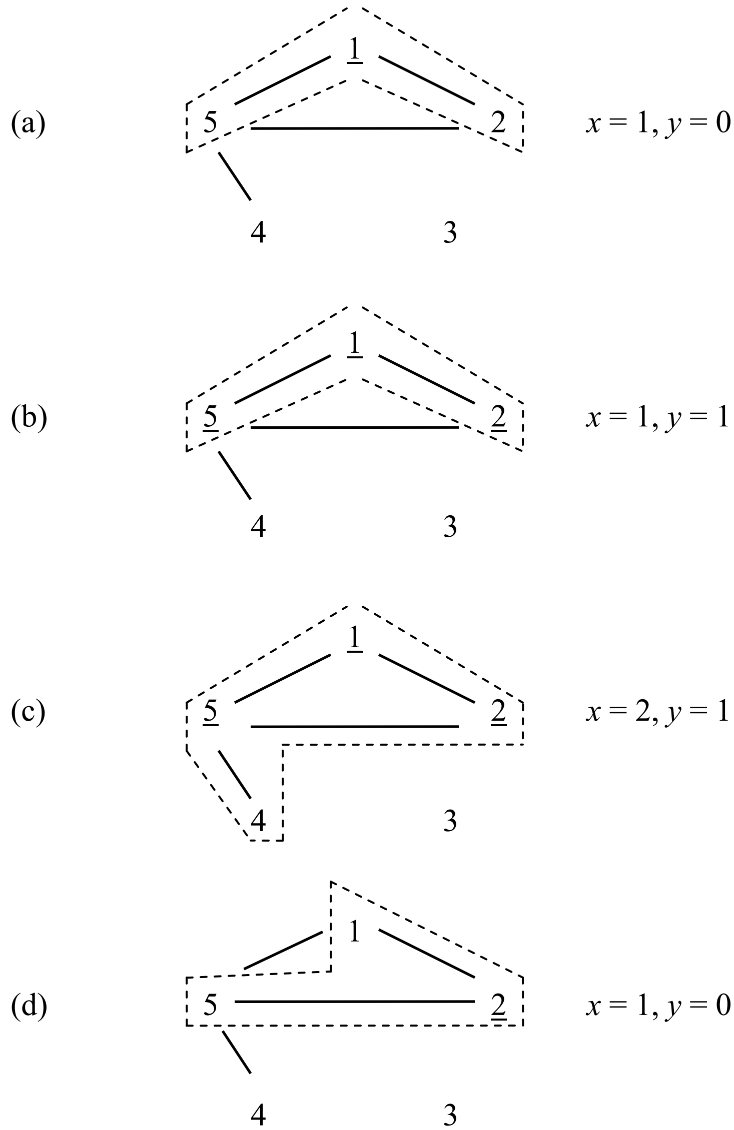
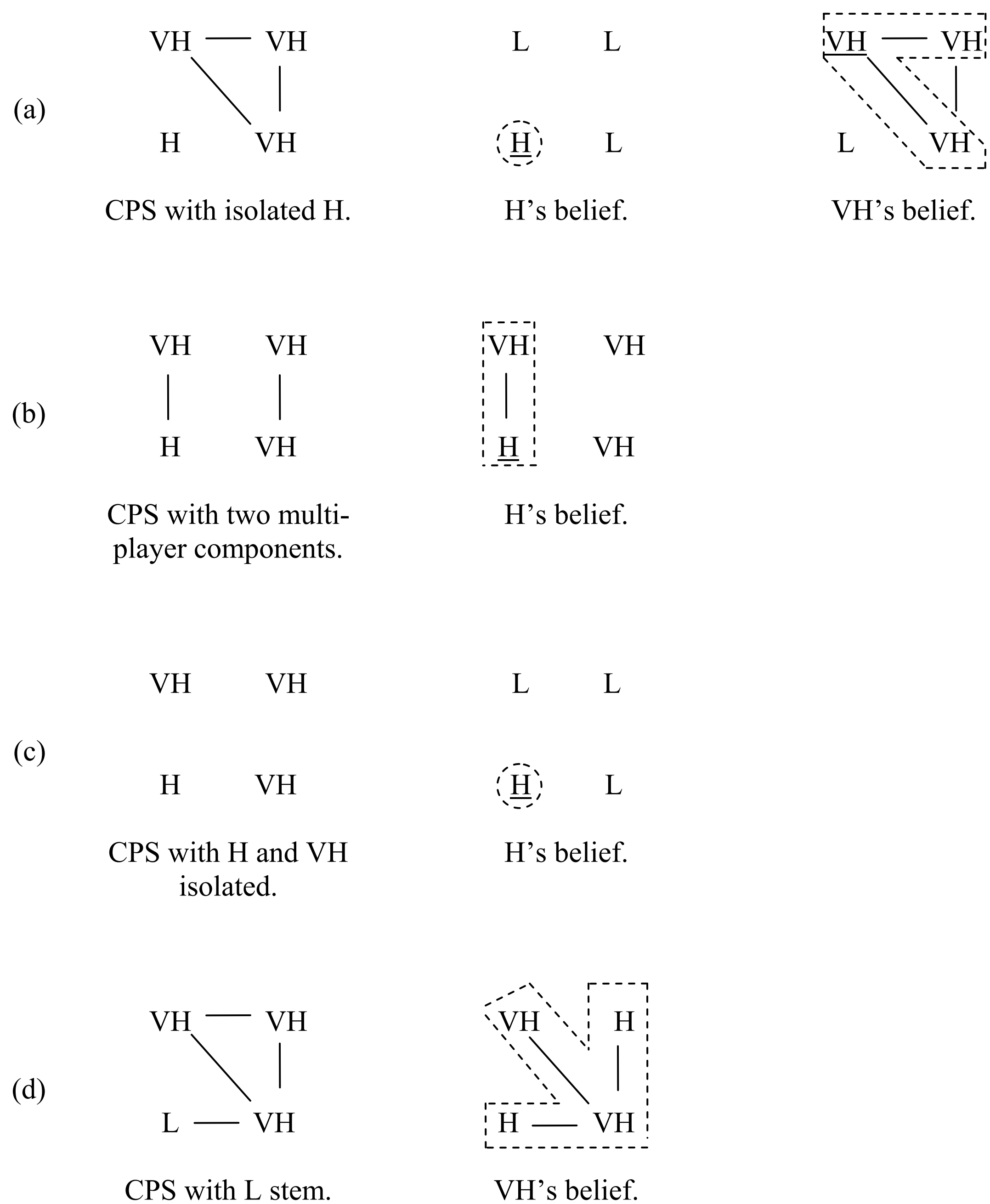


Figure 4


(a)
VH — VH
H VH
CPS with isolated H.
L L
H L
H's belief.
VH — VH
L VH
VH's belief.
(b)
VH VH
H VH
CPS with two multi-player components.
VH VH
H VH
H's belief.
(c)
VH VH
H VH
CPS with H and VH isolated.
L L
H L
H's belief.
(d)
VH — VH
L — VH
CPS with L stem.
VH H
H — VH
VH's belief.